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ORIGINAL PAPER

The dynamics of the stochastic multi-molecule biochemical reaction model

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Abstract The paper introduces the dynamics of a stochastic multi-molecule biochemical reaction model.First, we show that there is a unique positive solution of the stochastic model. Furthermore, we deduce the conditions when the reaction will end and when the reaction being proceed. At last, we derive that the solution of (1.5) oscillates around the endemic proportion equilibrium $P^*(x^*, y^*)$, and the intensity of fluctuation is proportional to white noise. The key to the analysis in this paper is choosing appropriate Lyapunov function. The outcomes are illustrated by computer simulations throughout this paper.

Keywords Stochastic chemical reaction model · Lyapunov function · Asymptotic behavior

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1 Introduction

In this paper,we will introduce a stochastic multi-molecule biochemical reaction model. In [1], E.E.Sel'kov describes a kinetic model of an open monosubstrate enzyme reaction with substrate inhibition and product activation. The reaction process is simplified as

$$[A_0]\underset{\rightarrow}{k_1} A_1,\ A_1 \underset{\rightarrow}{k_2} Q$$
$$pA_1 + qA_2 \underset{\rightarrow}{k_3} (p+q)A_2,\ A_2 \underset{\rightarrow}{k_4} P$$

Note the concentrations at time t of A_1 and A_2 are $x(t)$ and $y(t)$ separately. Then according to the law of mass action and the law of mass conservation, modelling the mathematical model of multi-molecule biochemical reaction is:

$$\begin{cases} \dfrac{dx}{dt} = k_1 x_0 - k_2 x - pk_3 x^p y^q \\ \dfrac{dy}{dt} = pk_3 x^p y^q - k_4 y \end{cases} \tag{1.1}$$

thereinto x_0 is the concentration of A_0, being set as constant. Through the dimensionless transformation, system (1.1) change into:

$$\begin{cases} \dfrac{dx_1}{dt} = \delta - ax_1 - {x_1}^p {x_2}^q \\ \dfrac{dx_2}{dt} = {x_1}^p {x_2}^q - bx_2 \end{cases} \tag{1.2}$$

where $x_1 \geq 0, x_2 \geq 0, a \geq 0, b > 0, \delta \geq 0$, p and q are positive integers. Some people research the dynamics of the system (1.2) under the assumption $a = 0$. In [2], author show that as $p = n, q = 2$ system (1.2) can produce stable limit cycles from Hopf bifurcations. The case of $p = 1$ is discussed perfectly in [3], and the existence of closed orbits is discussed in [4] when $p = 2$. But the whole research of this model has not yet been see, especially the situation of $a \neq 0$, while the model will appear more positive equilibrium points and multiple limit cycles.

Obviously, system (1.1) is similar to the famous epidemic model(SIS) with nonlinear incidence rate. If $p = q = 1$, then system (1.1) is similar to the SIS model which has bilinear incidence rate. If $0 < q < 1$ holds, using the same statement as in [5], we draw a conclusion, i.e. the system has two equilibrium states: an equilibrium Q_0 with the coordinates $\bar{x}_0 = \frac{k_1 x_0}{k_2}$, $\bar{y}_0 = 0$ and an endemic equilibrium state $Q^* = (x^*, y^*)$, such that

$$k_1 x_0 = k_2 x^* + pk_3 x^{*p} y^{*q},\ pk_3 x^{*p} y^{*q} = k_4 y^*.$$

In [5], the authors affirm that if $q \leq 1$, then the endemic equilibrium state Q^* of the system are globally asymptotically stable. Furthermore, the stability does not depend on the value of the parameter p.

Due to the model of oscillating chemical reactions requires p and q are positive integers, so in this paper, we consider the system (1.1) for $q = 1$ and $p \geq 1$, when system (1.1) become the following form:

$$\begin{cases} \dfrac{dx}{dt} = k_1x_0 - k_2x - pk_3x^p y \\ \dfrac{dy}{dt} = pk_3x^p y - k_4 y \end{cases} \tag{1.3}$$

Obviously, system (1.3) always has the boundary equilibrium $\bar{P} = (\bar{x}, \bar{y}) = (\frac{k_1x_0}{k_2}, 0)$. The threshold of system (1.3) is $R_0 = \frac{pk_3}{k_4}\left(\frac{k_1x_0}{k_2}\right)^p$. If $R_0 \leq 1$, then $\bar{P}$ is the unique equilibrium of (1.3) and it is globally stable in invariant set Γ, where $\Gamma = \{(x, y) : x > 0, y \geq 0, x + y \leq \frac{k_1x_0)}{k}\}$, here $k = min\{k_2, k_4\}$. If $R_0 > 1$, then $\bar{P}$ is unstable and there are endemic equilibrium $P^* = (x^*, y^*) = \left(\sqrt[p]{\frac{k_4}{pk_3}}, \frac{1}{k_4}(k_1x_0 - k_2\sqrt[p]{\frac{k_4}{pk_3}})\right)$ of (1.3), which is globally asymptotically stable under a sufficient condition in invariant set Γ.

In the process to establish the mathematical model of chemical reactions, for modelling, we usually assuming that the reaction rate is a constant under the premise that the reaction is carried out in the determined temperature and pressure. But in fact, the chemical reaction rate is closely related with the temperature and pressure when the reaction is proceeding. In addition, the chemical reaction rate is also affected by such as the catalyst, condition of concentration, solvent and other factors. It is apparently that chemical reaction models are inevitably affected by environmental white noise which is an important component in realism, because it can provide an additional degree of realism in compared to their deterministic counterparts.

Both from a chemical and from a mathematical perspective,there are different possible approaches to include random effects in the model. Now, let us consider the second equation of (1.3). To establish the stochastic differential equation(SDE) model, we naturally re-write the equation in the form of differential

$$dy(t) = \left[pk_3x^P(t)y(t) - k_4y(t)\right]dt \tag{1.4}$$

Here $[t, t + \triangle t)$ is a small time interval and we use the notation $d\cdot$ for the small change, for example $dy(t) = y(t + dt) - y(t)$ and the change $dy(t)$ is described by (1.4). Consider the reaction rate constant pk_3 in the deterministic model. This can be thought of the rate of reactant A_1 generation A_2, where A_1 collided with A_2 will cause the reaction proceeds. In the reaction process of

$$pA_1 + qA_2 \underset{\rightarrow}{k_3} (p + q)A_2$$

the total number of newly increased concentration of molecule A_2 in the small time interval $[t, t + dt)$ is

$$pk_3x^p y^q dt$$

and a unit concentration reactant A_1 makes

$$pk_3dt$$

A_2 collided with each other molecule in the small interval $[t, t + dt)$.

Now suppose that some stochastic environment factors acts simultaneously on each molecule in the reaction. In this case, pk_3 changes to a random variable $\tilde{k}$. More precisely, each A_1 generates

$$\tilde{k}dt = pk_3dt + \sigma dB(t)$$

A_2 in $[t, t+dt)$. Here $dB(t) = B(t+dt)-B(t)$ is the increment of a standard Brownian motion. Thus the concentration of newly increasing A_2 that a single A_1 collided with A_2 in $[t; t + dt)$ is normally distributed with mean pk_3dt and variance $\sigma^2 dt$. Hence $E(\tilde{k}dt) = pk_3dt$ and $var(\tilde{k}dt) = \sigma^2 dt$. As $var(\tilde{k}dt) \to 0$ as $dt \to 0$ this is a biologically reasonable model. Indeed this is a well-established way of introducing stochastic environmental noise into realistic chemical reaction dynamic models. See [6–12] and many other references.

Therefore we replace pk_3dt in Eq. (1.3) by $\tilde{k}dt = pk_3dt + \sigma dB(t)$ to get

$$dy(t) = [pk_3x^p(t)y(t) - k_4y(t)]dt + \sigma x^p(t)y(t)dB(t)$$

Note that $\tilde{k}dt$ now denotes the mean of the stochastic concentration of A_1 generate A_2 in the infinitesimally small time interval $[t, t + dt)$. Similarly, the first equation of (1.3) becomes another SDE. That is, the deterministic multi-molecule reaction model (1.3) becomes the $It\hat{o}$ SDE

$$\begin{cases} dx = (k_1x_0 - k_2x - pk_3x^py)dt - \sigma x^pydB(t) \\ dy = (pk_3x^py - k_4y)dt + \sigma x^pydB(t) \end{cases} \tag{1.5}$$

We will try to discuss the dynamics behavior of the system (1.5), which can easily determine the extinction and persistence of the reaction. This paper is organized as follows. In Sect. 2, we show there is a unique positive solution of system (1.5) by the same way as mentioned in Refs. [13–15]. In Sect. 3, we deduce the condition which will bring the reaction end. The condition for the reaction being persistent is given in Sect. 4. In Sect. 5, when $R_0 > 1$, we derive that the solution of (1.5) oscillates around the endemic proportion equilibrium $P^*(x^*, y^*)$, and the intensity of fluctuation is proportional to white noise. The key to the analysis in this paper is choosing appropriate Lyapunov function. Throughout the paper, outcomes of numerical simulations are reported in support of analytical results.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e.it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Denote

$$R^n_+ = \{x \in R^n : x_i > 0 \ for\ all\ 1 \leq i \leq n\}, \bar{R}^n_+ = \{x \in R^n : x_i \geq 0\ for\ all\ 1 \leq i \leq n\}$$

In general, consider d-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t),\ for\ t \geq t_0 \tag{1.6}$$

with initial value $x(t_0) = x_0 \in R^n$, $B(t)$ denotes d-dimensional standard Brownian motions defined on the above probability space. Define the differential operator L associated with Eq. (1.6) by

$$L = \frac{\partial}{\partial t} + \sum f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum \left[g^T(x,t)g(x,t)\right]_{ij} \frac{\partial^2}{\partial x_i x_j}$$

If L acts on a function $V \in C^{2,1}(R^n \times \overline{R}_+; \overline{R}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace[g^T(x,t)V_{xx}(x,t)g(x,t)]$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i x_j})_{d\times d}$. By It$\hat{o}$'s formula,if $x(t) \in S_h$, then

$$dV(x(t),t) = LV(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t).$$

Consider Eq. (1.6), assume $f(0,t) = 0$ and $g(0,t) = 0$ for all $t \geq t_0$. So $x(t) \equiv 0$ is a solution of Eq. (1.6), called the trivial solution or equilibrium position.

2 Existence and uniqueness of the positive solution

In this section we first show that the solution of system (1.5) is positive and global. To get a unique global (i.e.no explosion in a finite time) solution for any initial value, the coefficients of the equation are required to satisfy the local lipschitz condition and the linear growth condition (cf. Mao [16]). However, the coefficients of system (1.5) do not satisfy the linear growth condition, as the item $x^p y$ is nonlinear. So the solution of system (1.5) may explore in finite time. In this section, we use the Lyapunov analysis method, as mentioned in Refs. [13–15], to show that the solution of system (1.5) is positive and global.

Theorem 2.1 *There is a unique solution* $(x(t), y(t))$ *of system* (1.5) *on* $t \geq 0$ *for any initial value* $(x(0), y(0)) \in R^2_+$*, and the solution will remain in* R^2_+ *with probability 1, namely,* $(x(t), y(t)) \in R^2_+$ *for all* $t \geq 0$ *almost surely.*

Proof Since the coefficients of equation are locally Lipschitz continuous for any given initial value $(x(0), y(0)) \in R^2_+$, there is a unique local solution $(x(t), y(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (see Ref. [16]). To show this solution is global, we need to proof that $\tau_e = \infty a.s.$ Let $m_0 \geq 0$ be sufficiently large so that $x(0)$ and $y(0)$ all lie within the interval $[1/m_0, m_0]$. For each $m \geq m_0$, define the stopping time

$$\tau_m = inf\{t \in [0, \tau_e) : min\{x(t), y(t)\} \leq \frac{1}{m}\ or\ \max\{x(t), y(t)\} \geq m\}$$

where throughout this paper, we set $inf\phi = \infty$ (as usual ϕ denotes the empty set). According to the definition, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, where $\tau_\infty \leq \tau_e$ a.s. If we can prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(x(t), y(t)) \in R^2_+$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ $a.s.$ If not, there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon, \ for\ all \quad m \geq m_1. \tag{2.1}$$

For $t \leq \tau_m$, we can see, for each m,

$$d(x+y) = (k_1x_0 - k_2x - k_4y)dt \leq [k_1x_0 - k(x+y)]dt$$

where $k = min\{k_2, k_4\}$. Therefore,

$$x(t) + y(t) \leq \begin{cases} \frac{k_1x_0}{k}, & if\ x(0)+y(0) \leq \frac{k_1x_0}{k} \\ x(0)+y(0), & if\ x(0)+y(0) > \frac{k_1x_0}{k} \end{cases} := M$$

Define a C^2-function $V : R^2_+ \to \overline{R}_+$ by

$$V(x, y) = x - 1 - logx + (y - 1 - logy).$$

The non-negativity of this function can be seen from $u - 1 - logu \geq 0, \forall u > 0$. Let $m \geq m_0$ and $T > 0$ be arbitrary. Using Itô's formula, we obtain

$$\begin{aligned} dV(x, y) &= \left(1 - \frac{1}{x}\right) dx + \frac{1}{2x^2}(dx)^2 + \left(1 - \frac{1}{y}\right) dy + \frac{1}{2y^2}(dy)^2 \\ &= LVdt + \sigma(y - x)x^{p-1}dB(t). \end{aligned} \tag{2.2}$$

where $LV : R^2_+ \to \bar{R}_+$ is defined by

$$\begin{aligned} LV &= \left(1-\frac{1}{x}\right)(k_1x_0-k_2x-pk_3x^py)+\frac{\sigma^2}{2}x^{2(p-1)}y^2+\left(1-\frac{1}{y}\right)(pk_3x^py-k_4y) \\ &\quad +\frac{\sigma^2}{2}x^{2p} \\ &= k_1x_0+k_2+k_4-k_2x-\frac{k_1x_0}{k}-k_4y+pk_3x^{p-1}(x+y)+\frac{\sigma^2}{2}x^{2(p-1)}(y^2+x^2) \\ &\leq k_1x_0 + k_2 + k_4 + pk_3M^p + \sigma^2M^{2p} := C \end{aligned}$$

The remainder of the proof follows that in Ji et al. [17]. □

Remark 2.1 From Theorem 2.1 for any initial value $(x(0), y(0)) \in R^2_+$, there is a unique global solution $(x(t), y(t)) \in R^2_+$ almost surely of system (1.5). Hence

$$d(x+y) \le [k_1x_0 - k(x+y)]dt,$$

and

$$x(t) + y(t) \le \frac{k_1x_0}{k} + e^{-kt}\left[x(0) + y(0) - \frac{k_1x_0}{k}\right]$$

If $x(0) + y(0) \le \frac{k_1x_0}{k}$, then $x(t) + y(t) \le \frac{k_1x_0}{k}$ a.s. So the region

$$\Gamma^* = \left\{(x, y) \in R^2_+, x + y \le \frac{k_1x_0}{k}\right\} \tag{2.3}$$

is a positively invariant set of system (1.5), which is similar to Γ of system (1.3).

From now on, we always assume that $(x(0), y(0)) \in \Gamma^*$. For convenience we introduce the notation; let

$$\langle x(t)\rangle = \frac{1}{t}\int_0^t x(r)dr.$$

3 Conditions of the end of the reaction

In this section, we investigate the conditions for the extinction of the reaction.

Theorem 3.1 *Let* $(x(t), y(t))$ *be the solution of system* (1.5) *with initial value* $(x(0), y(0)) \in \Gamma^*$. *If*

(a) $\sigma^2 > \frac{pk_3}{2\bar{x}^p}$, *or*
(b) $R_0 - 1 < \frac{\sigma^2\bar{x}^{2p}}{2k_4}$ *and* $\sigma^2 \le \frac{pk_3}{2\bar{x}^p}$,

Then

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \le -k_4 + \frac{p^2{k_3}^2}{2\sigma^2} < 0 \text{ a.s. If (a) holds;} \tag{3.1}$$

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \le k_4(R_0 - 1 - \frac{\sigma^2\bar{x}^{2p}}{2k_4}) < 0 \text{ a.s. If (b) holds;} \tag{3.2}$$

(namely, $y(t)$ *tends to zero exponentially a.s. i.e., the reaction will end with probability* 1*). In addition,*

$$\lim_{t\to\infty} x(t) = \frac{k_1x_0}{k_2} = \bar{x}. \tag{3.3}$$

Proof An integration of system (1.5) yields

$$\begin{cases} \frac{x(t)-x(0)}{t} = k_1x_0 - k_2\langle x(t)\rangle - pk_3\langle x^p(t)y(t)\rangle - \frac{\sigma}{t}\int\limits_0^t x^p(r)y(r)dB(r) \\ \frac{y(t)-y(0)}{t} = pk_3\langle x^p(t)y(t)\rangle - k_4\langle y(t)\rangle + \frac{\sigma}{t}\int\limits_0^t x^p(r)y(r)dB(r). \end{cases} \tag{3.4}$$

According to (3.4), we have

$$\frac{x(t)-x(0)}{t} + \frac{y(t)-y(0)}{t} = k_1x_0 - k_2\langle x(t)\rangle - k_4\langle y(t)\rangle, \tag{3.5}$$

we compute that

$$\langle x(t)\rangle = \frac{k_1x_0}{k_2} - \frac{k_4}{k_2}\langle y(t)\rangle + \varphi(t), \tag{3.6}$$

where $\varphi(t)$ is defined by

$$\varphi(t) = -\frac{1}{k_2}\left[\frac{x(t)-x(0)}{t} + \frac{y(t)-y(0)}{t}\right].$$

Note (2.3), so

$$\lim_{t\to\infty}\varphi(t) = 0. \tag{3.7}$$

Applying It$\hat{o}$ formula to system (1.5) leads to

$$d(\log y) = \left(pk_3x^p - k_4 - \frac{\sigma^2}{2}x^{2p}\right)dt + \sigma x^p dB(t)$$

Integrating from 0 to t and dividing t on both sides, we have

$$\begin{aligned} \frac{\log y(t) - \log y(0)}{t} &= pk_3\langle x^p(t)\rangle - k_4 - \frac{\sigma^2}{2}\langle x^{2p}(t)\rangle + \frac{\sigma}{t}\int\limits_0^t x^p(r)dB(r) \\ &\le pk_3\langle x^p(t)\rangle - k_4 - \frac{\sigma^2}{2}\langle x^p(t)\rangle^2 + \frac{\sigma}{t}\int\limits_0^t x^p(r)dB(r) \\ &:= f(z) + \frac{M_1(t)}{t}. \end{aligned} \tag{3.8}$$

where $f : (0, (\frac{k_1x_0}{k_2})^p) \to R$ is defined by

$$f(z) = pk_3z - k_4 - \frac{\sigma^2}{2}z^2 = -\frac{\sigma^2}{2}\left(z - \frac{pk_3}{\sigma^2}\right)^2$$

$$+\frac{p^2 {k_3}^2}{2\sigma^2} - k_4, z = \langle x^p(t)\rangle \in \left[0, \left(\frac{k_1 x_0}{k_2}\right)^p\right]. \tag{3.9}$$

and

$$M_1(t) := \sigma \int_0^t x^p(r) dB(r). \tag{3.10}$$

which is a local continuous martingale and $M_1(0) = 0$. Moreover,

$$\limsup_{t\to\infty} \frac{\langle M_1, M_1\rangle_t}{t} \le \sigma^2 \left(\frac{k_1 x_0}{k}\right)^p < \infty \quad a.s.$$

By the large number theorem for martingales (see e.g. [16]), we obtain

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \ a.s. \tag{3.11}$$

If $\sigma^2 > \frac{pk_3}{2\bar{x}^p}$, (i.e. $\frac{pk_3}{2\sigma^2} < (\frac{k_1 x_0}{k_2})^p$), by (3.9), it is easy to see that

$$f(z) \le f\left(\frac{pk_3}{\sigma^2}\right) = \frac{p^2 {k_3}^2}{2\sigma^2} - k_4,$$

then from (3.8), we have

$$\frac{\log y(t)}{t} \le \frac{\log y(0)}{t} + f(z) + \frac{M_1(t)}{t} \le \frac{\log y(0)}{t} + \frac{p^2 {k_3}^2}{2\sigma^2} - k_4 + \frac{M_1(t)}{t}$$

Taking the limit superior of both sides, we obtain the desired assertion (3.1)

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \le \frac{p^2 {k_3}^2}{2\sigma^2} - k_4 < 0 \ a.s.$$

If $\sigma^2 \le \frac{pk_3}{2\bar{x}^p}$, (i.e. $\frac{pk_3}{2\sigma^2} \ge (\frac{k_1 x_0}{k_2})^p$), then

$$f(z) \le -\frac{\sigma^2}{2}\left(\left(\frac{k_1 x_0}{k_2}\right)^p - \frac{pk_3}{\sigma^2}\right)^2 + \frac{p^2 {k_3}^2}{2\sigma^2} - k_4,$$

from (3.8), we have assertion (3.2)

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \le k_4\left(R_0 - 1 - \frac{\sigma^2 \bar{x}^{2p}}{2k_4}\right) < 0 \ a.s.$$

which implies $\lim_{t\to\infty} y(t) = 0$ a.s.

Next, we prove the assertion (3.3). According to system (1.5), we obtain

$$d(x+y)=(k_1x_0-k_2x-k_4y)dt=[k_1x_0-k_2(x+y)+(k_2-k_4)y]dt \tag{3.12}$$

From (3.12) we can formally solve to obtain

$$x(t)+y(t)=e^{-k_2t}\{x(0)+y(0)+\int_0^t[k_1x_0-(k_4-k_2)y(s)]e^{k_2s}ds\}$$

Applying L'Hospital's rule and (3.11), we get

$$\begin{aligned}\lim_{t\to\infty}x(t)&=\lim_{t\to\infty}\left\{\frac{x(0)+y(0)+\int_0^t[k_1x_0-(k_4-k_2)y(s)]e^{k_2s}ds}{e^{k_2t}}-y(t)\right\}\\&=\lim_{t\to\infty}\frac{k_1x_0-(k_4-k_2)y(t)}{k_2}=\frac{k_1x_0}{k_2}=\bar{x}.\end{aligned}$$

This finish the proof. □

Remark 3.1 Theorem 3.1 tells us the reaction will end if $R_0-1<\frac{\sigma^2\bar{x}^{2p}}{2k_4}$, and the white noise is not large. While if the white noise is large enough such that $\sigma^2>\frac{pk_3}{2\bar{x}^p}$ is satisfied, then the reaction will also end. The following example illustrate this result more explicitly.

Example 3.1 Throughout the paper we shall assume that the unit of time is minute and the concentrations of the reactant are measured in units of $mol/L\cdot min$. Choosing the parameters in the system (1.5) as follows:

$$k_1=1.4,\ x_0=1,\ k_2=1.2,\ k_3=0.5,\ k_4=1.2,\ \sigma=0.6, \tag{3.13}$$

Here we choose $p=2$, that is one of the conditions of $p\geq 1$. Note that

$$R_0-\frac{\sigma^2\bar{x}^{2p}}{2k_4}\doteq 0.856<1$$

and

$$\sigma^2=0.36<\frac{pk_3}{2\bar{x}^p}\doteq 0.367$$

then by Theorem 3.1, the solution $(x(t),y(t))$ of system (1.5) obeys

$$\limsup_{t\to\infty}\frac{\log y(t)}{t}\leq k_4\left(R_0-1-\frac{\sigma^2\bar{x}^{2p}}{2k_4}\right)=-0.1728<0,\quad a.s.$$

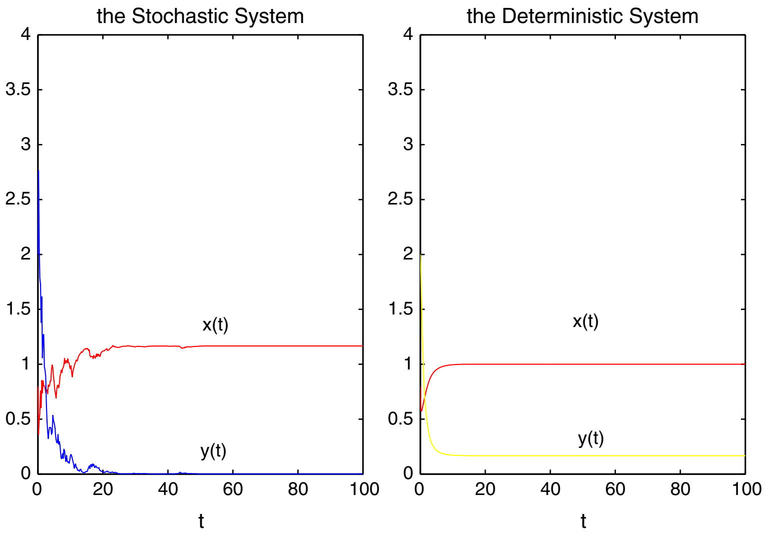

Fig. 1 Computer simulation of the path $x(t)$, $y(t)$ for the SDE multi-molecule reaction model (1.5) and its corresponding deterministic model (1.3) for parameter values $k_1 = 1.4, x_0 = 1, k_2 = 1.2, k_3 = 0.5, k_4 = 1.2, \sigma = 0.6$ and $p = 2$. Using the EM method with step size $\triangle t = 0.001$ and initial value $(x(0), y(0)) = (0.8, 2)$

and

$$\lim_{t\to\infty} x(t) = \frac{k_1 x_0}{k_2} = \bar{x} \doteq 1.167,$$

with any initial value $(x(0), y(0)) = (0.8, 2) \in \Gamma^*$. That is $y(t)$ will tends to zero exponentially with probability one. Otherwise, for the corresponding deterministic model (1.3)

$$R_0 = \frac{pk_3}{k_4}\left(\frac{k_1 x_0}{k_2}\right)^p \doteq 1.134 > 1;$$

then the endemic equilibrium (x^*, y^*) is globally asymptotically stable in Γ. Using the method mentioned in [18], we give the simulations shown in Fig. 1 to support our results.

Example 3.2 We keep all the parameters of (3.13) unchanged but increase σ to 0.9. Note that $\sigma^2 > \frac{pk_3}{2\bar{x}^p} \doteq 0.367$; then by Theorem 3.1, the solution $(x(t), y(t))$ of system (1.5) obeys

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \leq -k_4 + \frac{p^2 k_3{}^2}{2\sigma^2} \doteq 0.5827 < 0 a.s.$$

That is $y(t)$ will tend to zero exponentially with probability one. But for the corresponding deterministic model (1.3), $R_0 > 1$; then the endemic equilibrium (x^*, y^*)

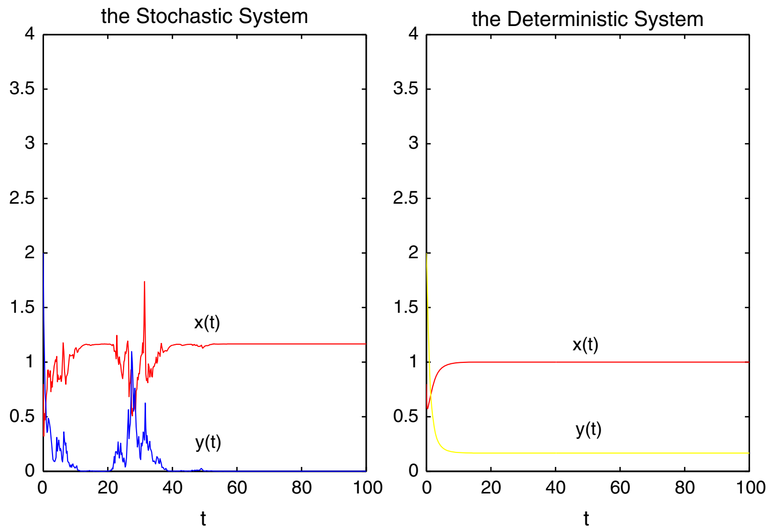

Fig. 2 Computer simulation of the path $x(t)$, $y(t)$ for the SDE multi-molecule reaction model (1.5) and its corresponding deterministic model (1.3) for parameter values $k_1 = 1.4, x_0 = 1, k_2 = 1.2, k_3 = 0.5, k_4 = 1.2, \sigma = 0.9$ and $p = 2$. Using the EM method with step size $\triangle t = 0.001$ and initial value $(x(0), y(0)) = (0.8, 2)$

is globally asymptotically stable in Γ. Using the method mentioned in Ref. [18], we give the simulation shown in Fig. 2 to sustain our results.

4 Continuous reaction conditions

Definition 4.1 System (1.5) is said to be persistence in the mean, if

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(r)dr > 0 a.s.$$

Theorem 4.1 *If*

$$\tilde{R_0} = R_0 - \frac{\sigma^2}{2k_4}\left(\frac{k_1 x_0}{k}\right)^{2p} > 1, \tag{4.1}$$

then for any initial value $(x(0), y(0)) \in \Gamma^*$*, the solution* $(x(t), y(t))$ *of system* (1.5) *has the following property:*

$$\liminf_{t\to\infty} \langle y(t)\rangle \geq \tilde{y}.\ a.s. \tag{4.2}$$

where

$$\tilde{y} = -\frac{k_2}{k_4}\left(\frac{k_4}{pk_3} + \frac{\sigma^2}{2pk_3}\left(\frac{k_1 x_0}{k}\right)^p\right)^{\frac{1}{p}} + \frac{k_1 x_0}{k_4} > 0. \tag{4.3}$$

Proof By the same way as the Proof of Theorem 3.1, we have

$$\frac{\log y(t)-\log y(0)}{t}=pk_3\langle x^p(t)\rangle-k_4-\frac{\sigma^2}{2}\langle x^{2p}(t)\rangle+\frac{M_1(t)}{t},$$

where $\langle x(t)\rangle$ is computed by (3.6), and the definition of $M_1(t)$ is similar to (3.10). By the property of $\langle x(t)\rangle$, i.e.$\langle x^p(t)\rangle \geq \langle x(t)\rangle^p$, we have the following inequality:

$$\begin{aligned}\frac{\log y(t)-\log y(0)}{t} &\geq pk_3\langle x(t)\rangle^p-k_4-\frac{\sigma^2}{2}\left(\frac{k_1x_0}{k}\right)^{2p}+\frac{M_1(t)}{t}\\ &\geq pk_3\left[\frac{k_1x_0}{k_2}-\frac{k_4}{k_2}\langle y(t)\rangle+\varphi(t)\right]^p-k_4-\frac{\sigma^2}{2}\left(\frac{k_1x_0}{k}\right)^{2p}+\frac{M_1(t)}{t}.\end{aligned}$$

Note that $0<x+y<\frac{k_1x_0}{k}$, we have $-\infty<logy(t)<log\frac{k_1x_0}{k}$. Thus we get the following inequality from the above formula:

$$\frac{\log\frac{k_1x_0}{k}-\log y(0)}{t}\geq pk_3\left[\frac{k_1x_0}{k_2}-\frac{k_4}{k_2}\langle y(t)\rangle+\varphi(t)\right]^p-k_4-\frac{\sigma^2}{2}\left(\frac{k_1x_0}{k}\right)^{2p}+\frac{M_1(t)}{t}. \tag{4.4}$$

Rearrange the inequality (4.4) we can get

$$\begin{aligned}\langle y(t)\rangle \geq & -\frac{k_4}{k_2}\left\{\frac{1}{pk_3}\left[k_4+\frac{\sigma^2}{2}\left(\frac{k_1x_0}{k}\right)^{2p}+\frac{\log y(t)-\log y(0)}{t}+\frac{M_1(t)}{t}\right]\right\}^{\frac{1}{p}}\\ &+\frac{k_1x_0}{k_4}+\frac{k_2}{k_4}\varphi(t).\end{aligned} \tag{4.5}$$

By (3.7) and (3.11), then taking the limit inferior of both sides (4.5) leads to

$$\liminf_{t\to\infty}\langle y(t)\rangle \geq -\frac{k_2}{k_4}\left[\frac{k_4}{pk_3}+\frac{\sigma^2}{2pk_3}\left(\frac{k_1x_0}{k}\right)^{2p}\right]^{\frac{1}{p}}+\frac{k_1x_0}{k_4}:=\tilde{y}.$$

Therefore, by the condition (4.1), we have the assertion (4.2) and (4.3). This complete the proof of Theorem 4.1. □

Remark 4.1 Inducing Theorems 3.1 and 4.1, we can see when the noise is so small that $\sigma^2<\frac{pk_3}{2\bar{x}^p}$ the value of $R_{01}:=R_0-\frac{\sigma^2}{2k_4}\bar{x}^{2p}<1$ will lead to the reaction end, and the value of $R_{02}:=R_0-\frac{\sigma^2}{2k_4}\left(\frac{k_1x_0}{k}\right)^{2p}>1$ will lead to the reaction proceeds. Obviously $R_{01}>R_{02}$.

Example 4.1 Assume that the parameter of system (1.5) are given by

$$k_1=1.4, x_0=1, k_2=1.2, k_3=0.5, k_4=1.2, \sigma=0.2, \tag{4.6}$$

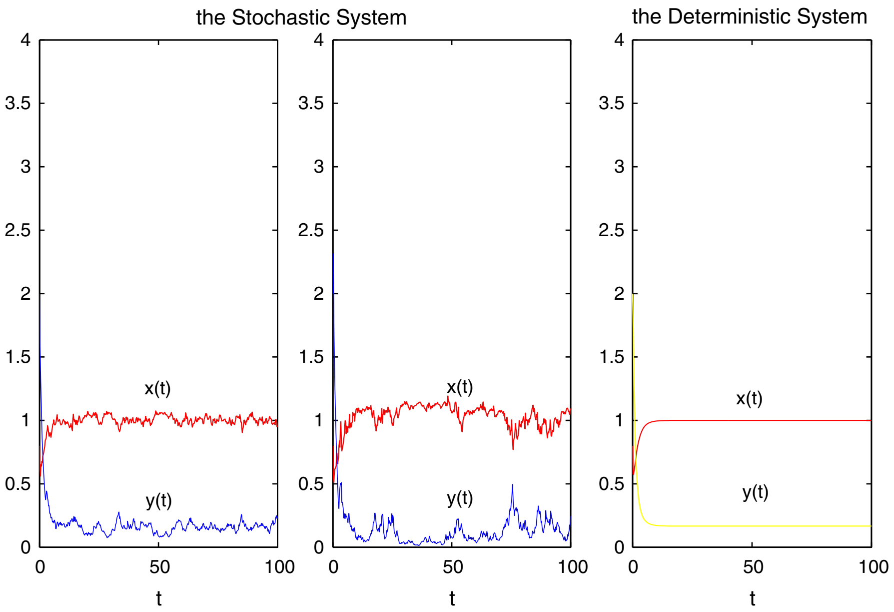

Fig. 3 Computer simulation of the path $x(t)$, $y(t)$ for the SDE multi-molecule reaction model (1.5) and its corresponding deterministic model (1.3) for parameter values $k_1 = 1.4$, $x_0 = 1$, $k_2 = 1.2$, $k_3 = 0.5$, $k_4 = 1.2$, $p = 2$, $\sigma = 0.2$ and $\sigma = 0.4$. Using the EM method with step size $\triangle t = 0.001$ and initial value $(x(0), y(0)) = (0.8, 2)$

Note that $R_0 - \frac{\sigma^2}{2k_4}\left(\frac{k_1 x_0}{k}\right)^{2p} \doteq 1.103 > 1$ and $\sigma^2 < \frac{pk_3}{2\bar{x}^p} \doteq 0.367$.

Then by the Theorem 4.1, for any initial value $(x(0), y(0)) = (0.8, 2) \in \Gamma^*$, we conclude that the solution $(x(t), y(t))$ of system (1.5) obeys

$$\liminf_{t\to\infty}\langle y(t)\rangle \geq -\frac{k_2}{k_4}\left(\frac{k_4}{pk_3} + \frac{\sigma^2}{2pk_3}\left(\frac{k_1 x_0}{k}\right)^p\right)^{\frac{1}{p}} + \frac{k_1 x_0}{k_4} \doteq 0.224 > 0, a.s.$$

That is to say, the reaction will proceed.

To further illustrate the effect of the noise intensity σ on model (1.5), we keep all the parameter of (4.6) unchanged but increase σ to 0.4. Note that

$$R_0 - \frac{\sigma^2}{2k_4}\left(\frac{k_1 x_0}{k}\right)^{2p} \doteq 1.01048 > 1.$$

Using the method mentioned in Ref. [18], we give the simulations to support our results in Fig. 3. Comparing the first picture and the second picture in Fig. 3, when the noise getting smaller, the fluctuation of the solution of system (1.5) is getting weaker.

5 Asymptotic behavior around the endemic proportion equilibrium $P^*(x^*, y^*)$

Theorem 5.1 *If $R_0 > 1$, then for any initial value $(x(0), y(0)) \in \Gamma^*$, the solution $(x(t), y(t))$ of system (1.5) has the following property:*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \left[k_2(x(r)-x^*)^2 + \frac{k_4}{2}(y(r)-y^*)^2\right]dr$$
$$\leq \left(\frac{k_1x_0}{k}\right)^{2p+1}\left\{\frac{(k_2+k_4)^2x^{*2}}{4k_4(k_2x^*+k_4y^*)}\left[\frac{k_4}{k_3}+\left(\frac{k_1x_0}{k}\right)^{p-1}y^*\right]+\frac{k_1x_0}{k}\right\}\sigma^2$$
$$a.s. \tag{5.1}$$

Proof Since $P^*(x^*, y^*)$ is the endemic equilibrium of system (1.5), we have

$$k_1x_0 = k_2x^* + pk_3x^{*p}y^*, k_4 = pk_3x^{*p}.$$

Define a C^2-function $V : (0, \frac{k_1x_0}{k}) \times (0, \frac{k_1x_0}{k}) \longrightarrow R_+$ by

$$V(x, y) = a\left\{\int_{x^*}^{x}\left[1-\left(\frac{x^*}{s}\right)^p\right]ds + \left(y - y^* - y^*log\frac{y}{y^*}\right)\right\}$$
$$+\frac{1}{2}(x - x^* + y - y^*)^2 := aV_1 + V_2$$

This function is nonnegative for all $x, y > 0$ because of the fact that $u - 1 - \log u \geq 0$ on $u > 0$. Let L be the generating operator of system (1.5), then we get

$$\begin{aligned}
LV_1 &= \left[1-\left(\frac{x^*}{x}\right)^p\right](k_1x_0 - k_2x - pk_3x^py) + \frac{\sigma^2}{2}px^{*P}x^{p-1}y^2\\
&\quad+\left(1-\frac{y^*}{y}\right)(pk_3x^py - k_4y) + \frac{\sigma^2}{2}x^{2p}y^*\\
&= k_1x_0 - k_2x - k_1x_0\left(\frac{x^*}{x}\right)^p + k_2x\left(\frac{x^*}{x}\right)^p + pk_3x^{*p}y - k_4y - pk_3x^py^* + k_4y^*\\
&\quad+\frac{\sigma^2}{2}px^{*p}x^{p-1}y^2 + \frac{\sigma^2}{2}x^{2p}y^*\\
&= k_2x^* - k_2x + k_2x\left(\frac{x^*}{x}\right)^p - k_2x^*\left(\frac{x^*}{x}\right)^p + k_4y^* - k_4y^*\left(\frac{x^*}{x}\right)^p + k_4y^*\left(\frac{x^*}{x}\right)^p\\
&\quad+\frac{\sigma^2}{2}px^{*p}x^{p-1}y^2 + \frac{\sigma^2}{2}x^{2p}y^*\\
&= k_2x^*(1-u)\left(1-\frac{1}{u^p}\right) + k_4y^*(u^p-1)\left(\frac{1}{u^p}-1\right) + \frac{\sigma^2}{2}px^{*p}x^{p-1}y^2 + \frac{\sigma^2}{2}x^{2p}y^*\\
&\leq -\frac{k_2x^*}{u^p}(u-1)(u^p-1) - \frac{k_4y^*}{u^p}(u^p-1)^2 + \frac{\sigma^2}{2}px^{*p}x^{p-1}y^2 + \frac{\sigma^2}{2}x^{2p}y^*.
\end{aligned}$$

Here $u = \frac{x}{x^*}$, If $p \geq 1$, then

$$h(u) = (u-1)(u^p - 1) \geq 0 \; for\, all\, u > 0 \tag{5.2}$$

and we can give a simple proof of the fact

$$|u^p - 1| \geq |u - 1|, \; p \geq 1. \tag{5.3}$$

It is obvious that if $u \geq 1$, then $u^p - 1 \geq 0$, $u - 1 \geq 0$ and we have $u^p - 1 \geq u - 1$; if $u < 1$, then $u^p - 1 < 0$, $u - 1 < 0$ and we have $u^p - 1 \leq u - 1$. Synthesize this two points, we can conclude that (5.3) is true. Then from (2.3), (5.2) and (5.3), we obtain

$$\begin{aligned} LV_1 &\leq -\frac{k_2x^* + k_4y^*}{\left(\frac{k_1x_0}{k}\right)^p}|u-1|^2 + \frac{\sigma^2}{2}px^{*p}x^{p-1}y^2 + \frac{\sigma^2}{2}x^{2p}y^* \\ &\leq -\left(\frac{k}{k_1x_0}\right)^p \frac{k_2x^*+k_4y^*}{x^{*2}}(x-x^*)^2 + \frac{\sigma^2}{2}\left(\frac{k_1x_0}{k}\right)^{p+1}\left[\frac{k_4}{k_3} + \left(\frac{k_1x_0}{k}\right)^{p-1} y^*\right], \end{aligned}$$

and

$$\begin{aligned} LV_2 &= (x - x^* + y - y^*)(k_1x_0 - k_2x - k_4y) + \sigma^2x^{2p}y^2 \\ &\leq -k_2(x-x^*)^2 - (k_2+k_4)(x-x^*)(y-y^*) - k_4(y-y^*)^2 + \sigma^2\left(\frac{k_1x_0}{k}\right)^{2(p+1)}. \end{aligned}$$

Then

$$\begin{aligned} LV &= aLV_1 + LV_2 \\ &\leq -\left[a\left(\frac{k}{k_1x_0}\right)^p \frac{k_2x^*+k_4y^*}{x^{*2}} + k_2\right](x-x^*)^2 - (k_2+k_4)(x-x^*)(y-y^*) - k_4(y-y^*)^2 \\ &\quad + a\frac{\sigma^2}{2}\left(\frac{k_1x_0}{k}\right)^{p+1}\left[\frac{k_4}{k_3} + \left(\frac{k_1x_0}{k}\right)^{p-1} y^*\right] + \sigma^2\left(\frac{k_1x_0}{k}\right)^{2(p+1)}. \end{aligned}$$

Using the Young's inequality we have

$$\begin{aligned} LV &\leq -\left[a\left(\frac{k}{k_1x_0}\right)^p \frac{k_2x^* + k_4y^*}{x^{*2}} + k_2 - \frac{(k_2+k_4)^2}{2k_4}\right](x-x^*)^2 - \frac{k_4}{2}(y-y^*)^2 \\ &\quad + a\frac{\sigma^2}{2}\left(\frac{k_1x_0}{k}\right)^{p+1}\left[\frac{k_4}{k_3} + \left(\frac{k_1x_0}{k}\right)^{p-1} y^*\right] + \sigma^2\left(\frac{k_1x_0}{k}\right)^{2(p+1)}. \end{aligned}$$

Furthermore we choose a such that $a(\frac{k}{k_1x_0})^p\frac{k_2x^*+k_4y^*}{x^{*2}} - \frac{(k_2+k_4)^2}{2k_4} = 0$, therefore

$$\begin{aligned} LV &\leq -k_2(x-x^*)^2 - \frac{k_4}{2}(y-y^*)^2 \\ &\quad + \left(\frac{k_1x_0}{k}\right)^{2p+1}\left\{\frac{(k_2+k_4)^2x^{*2}}{4k_4(k_2x^*+k_4y^*)}\left[\frac{k_4}{k_3} + \left(\frac{k_1x_0}{k}\right)^{p-1} y^*\right] + \frac{k_1x_0}{k}\right\}\sigma^2 := F(t). \end{aligned}$$

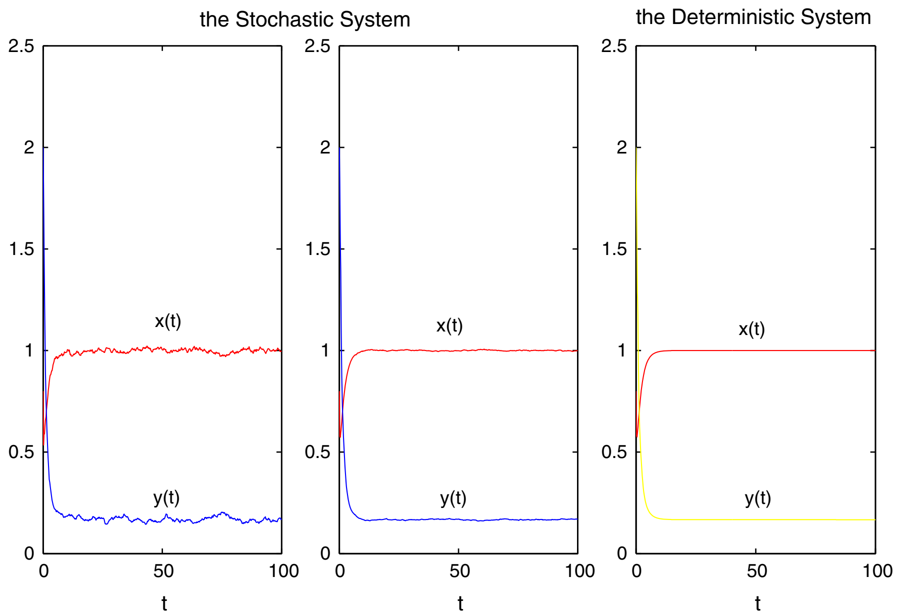

Fig. 4 Computer simulation of the path $x(t)$, $y(t)$ for the SDE multi-molecule reaction model (1.5) and its corresponding deterministic model (1.3) for parameter values $k_1 = 1.4$, $x_0 = 1$, $k_2 = 1.2$, $k_3 = 0.5$, $k_4 = 1.2$, $p = 2$ and differing values of $\sigma = 0.05$ and $\sigma = 0.01$. Using the EM method with step size $\triangle t = 0.001$ and initial value $(x(0), y(0)) = (0.8, 2)$

So $dV \leq F(t)dt + \sigma(x^{*p}y - x^p y^*)dB(t)$. Integrating both sides of it from 0 to t, yields

$$V(t) - V(0) \leq \int_0^t F(s)ds + \int_0^t \sigma(x^{*p}y - x^p y^*)dB(s). \tag{5.4}$$

Let $M_2(t) := \int_0^t \sigma(x^{*p}y - x^p y^*)dB(s)$. By the large number theorem for martingales (see e.g. [16]), we obtain $\lim_{t\to\infty} \frac{M_2(t)}{t} = 0$ a.s., which together with (5.4) implies

$$\limsup_{t\to\infty} \frac{\int_0^t F(s)ds}{t} \geq 0 \; a.s.$$

Consequently we get formula (5.1). This complete the proof of Theorem 5.1. □

Remark 5.1 Theorem 5.1 shows that under some conditions, the distance between the solution $X(t) = (x(t), y(t))$ of system (1.5) and the endemic proportion equilibrium $P^* = (x^*, y^*)$ of syetem (1.3) has the following form

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \|X(s) - P^*\|^2 \leq N\|\sigma\|^2, a.s.$$

where N is a positive constant. Although, the solution of system (1.5) does not have stability as the corresponding deterministic system, we can still think there is approximate stability, provided $\|\sigma\|^2$ is sufficiently small.

Example 5.1 Assume that the parameters of system (1.5) are given by

$$k_1 = 1.4, x_0 = 1, k_2 = 1.2, k_3 = 0.5, k_4 = 1.2, \sigma = 0.05.$$

That is, we keep all the parameters the same as in Example 3.2 but let $\sigma = 0.05$. Note that

$$R_0 = \frac{pk_3}{k_4}\left(\frac{k_1 x_0}{k_2}\right)^p \doteq 1.134 > 1,$$

then by Theorem 5.1, for any initial value $(x(0), y(0) \in (0, \frac{k_1 x_0}{k}) \times (0, \frac{k_1 x_0}{k})$, we draw a conclusion that the difference between the perturbed solution $(x(t), y(t))$ of system (1.5) and $P^* = (x^*, y^*)$ is only related with white noise under the condition $R_0 > 1$. Using the method shown in [16], we give the simulation to support our result. As expected, the solution is oscillating around the endemic equilibrium P^* for a long time (see Fig. 4). Besides, the parameters of the first two pictures in Fig. 4 are all same but with different intensities of white noise. Especially, in the first $\sigma = 0.05$ and in the second $\sigma = 0.01$. Obviously, we can observe when the white noise getting weaker, the fluctuation around P^* become smaller,which supports the result of Theorem 5.1.

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